◎ One Square System = Two Triangular Systems

$A\underline{x} = \underline{b}$

Suppose elimination requires no row exchanges.

$A = LU$

$\Rightarrow LU\underline{x} = \underline{b}$

$\Rightarrow U\underline{x} = L^{-1}\underline{b} = \underline{c}$

We have $U\underline{x} = \underline{c}$ where $L\underline{c} = \underline{b}$.

1. Factor $A = LU$ by Gaussian elimination $\underline{c} = L^{-1}\underline{b}$
2. Solve $\underline{c}$ from $L\underline{c} = \underline{b}$ (forward elimination) and then solve $U\underline{x} = \underline{c}$ (back elimination)

☆ Example:

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$A\underline{x} = \underline{b}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}$$

$$L\underline{c} = \underline{b} \quad (\text{or } \underline{c} = L^{-1}\underline{b})$$

$$\therefore \underline{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & -8 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -12 \\ 2 \end{bmatrix}$$

$$U\underline{x} = \underline{c}$$

$$\therefore x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

◎ Complexity of Elimination

① Solve $A\underline{x} = \underline{b}$

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = LU$$

Elimination : 1st stage : $n(n-1) \approx n^2$

(# multiplications and # additions)

2nd stage : $(n-1)^2$

3rd stage : $(n-2)^2$

$\vdots$

$$n^2 + (n-1)^2 + \cdots + 1^2 = \frac{1}{3}n(n+\frac{1}{2})(n+1)$$
$$\approx \frac{n^3}{3}$$

Solve $L\underline{c} = \underline{b}$

$$\begin{bmatrix} 1 & & & & 0 \\ \times & 1 & & & \\ \times & \times & \ddots & & \\ \times & \times & \cdots & \ddots & \\ \times & \times & \cdots & \cdots & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$$

$$(n-1) + (n-2) + \cdots + 2 + 1 = \frac{n(n-1)}{2} \approx \frac{n^2}{2}$$

$U\underline{x} = \underline{c}$

$$\begin{bmatrix} p_1 & \times & \times & \times & \times \\ & p_2 & \times & \cdots & \times \\ & & \ddots & & \times \\ & 0 & & \ddots & \times \\ & & & & p_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{bmatrix}$$

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2} \approx \frac{n^2}{2}$$

$$\# = \frac{n^2}{2} + \frac{n^2}{2} = n^2 \ll \frac{n^3}{3}$$

$$\text{Total } \# = \frac{n^3}{3}$$

② Compute A^{-1}

Solve $A[\underline{x_1}\ \underline{x_2} \cdots \underline{x_n}] = [\underline{e_1}\ \underline{e_2} \cdots \underline{e_n}]$

where $\underline{e_i} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ⟶ ith component = 1

Elimination : $A = LU$ $\quad \# = \frac{n^3}{3}$

Solve :

$$L\,\underline{c_i} = \underline{e_i} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow i$$

Need to solve $(n-i+1)$ variables

$$\# = \frac{(n-i+1)^2}{2}$$

Since $1 \leq i \leq n$, $\# = \frac{n^2}{2} + \frac{(n-1)^2}{2} + \cdots + \frac{2^2}{2} + \frac{1^2}{2} \approx \frac{n^3}{6}$

$U\,\underline{x} = \underline{c_i} \qquad \# = \frac{n^2}{2}$

Since $1 \leq i \leq n$, $\# = n \cdot \frac{n^2}{2} = \frac{n^3}{2}$

$\therefore$ Total $\# = \frac{n^3}{3} + \frac{n^3}{6} + \frac{n^3}{2} = n^3$

Compared with # for $A^2 = A \cdot A$: $n \cdot n^2 = n^3$.

◎ Transposes and Permutations

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix} \qquad \underset{\text{transpose of } A}{A^T} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$$

$$(A^T)_{ij} = A_{ji}$$

<u>Claim</u> $(A+B)^T = A^T + B^T$

<u>Claim</u> $(AB)^T = B^T A^T \qquad A : n \times m \qquad B : m \times \ell$

<u>Proof</u> $[(AB)^T]_{ij} = (AB)_{ji} = \sum_{k=1}^{m} A_{jk} B_{ki} = \sum_{k=1}^{m} (A^T)_{kj} (B^T)_{ik}$

$= \sum_{k=1}^{m} (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}$ ∎

<u>Remark</u> $(ABC)^T = C^T B^T A^T$

<u>Claim</u> $(A^{-1})^T = (A^T)^{-1}$

<u>Proof</u>
$$\left.\begin{array}{l} AA^{-1} = I \Rightarrow (A^{-1})^T A^T = I^T = I \\ A^{-1}A = I \Rightarrow A^T (A^{-1})^T = I^T = I \end{array}\right\} \Rightarrow (A^T)^{-1} = (A^{-1})^T$$ ∎

<u>Def</u> An $n \times n$ matrix A is symmetric if $A^T = A$.

<u>Remark</u> $A_{ij} = A_{ji}$ if A is symmetric.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A^T \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix} = D^T$$

<u>Claim</u> Given any matrix $\underset{n\times m}{R}$, $\underset{m\times n}{R^T}\underset{n\times m}{R}$ is symmetric.

<u>Proof</u> $(R^TR)^T = R^T(R^T)^T = R^TR$ ∎

<u>Remark</u> RR^T is also symmetric.

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} \text{ symmetric.}$$

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

transpose

$$A = LDL^T$$

<u>Claim</u> If a symmetric matrix is factored into LDU with no row exchanges, then $U = L^T$.

<u>Proof</u> $A = LDU \Rightarrow A^T = U^TD^TL^T = U^TDL^T$

Since A is symmetric, $A = LDU = U^TDL^T$

Recall that this factorization is unique. We then have $U = L^T$. ∎

<u>Def</u> A permutation matrix has the rows of the identity I in any order.

There are <u>6</u> 3×3 permutation matrices.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad P_{32}P_{21} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad P_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad P_{21}P_{32} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

∴ There are <u>$n!$</u> permutation matrices of order n.

<u>Claim</u> If P is a permutation matrix, then $P^{-1} = P^T$.

<u>Proof</u>

$$P = \begin{bmatrix} 0\,1\,0\,0 \cdots 0 \end{bmatrix} \qquad P^T = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

i th row $\qquad$ ith column

$(PP^T)_{ii} = 1$ for all i
$(PP^T)_{ij} = 0$ if $i \neq j$.
$\therefore PP^T = I$. ∎

Recall if no row exchanges are required, then $A = LU$.
If row exchanges are needed, we then have
$(\cdots P \cdots E \cdots P \cdots E)A = U$
$\Rightarrow A = (E^{-1} \cdots P^{-1} \cdots E^{-1} \cdots P^{-1} \cdots)U$

If row exchanges are needed during elimination, we can do them in advance. The product PA will put the rows in the right order so that no row exchanges are needed for PA. Hence $\boxed{PA = LU.}$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

x-2 x-3

$$\underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}$$

If we hold row exchanges until after elimination, we then have $\boxed{A = L_1 P_1 U_1}$